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Lagrange and Hermite Interpolation Processes on the Positive Real Line

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1. INTRODUCTION

In this paper we consider interpolation based on the Laguerre roots and the point 0 as nodes. First we show that this interpolation generates a convergent approximation process on $[0, \infty)$ for a wide class of functions. Moreover, we prove the following interesting fact: In order to have uniform convergence of the derivatives of the interpolating polynomials in every interval $[0, A]$, it is sufficient to prescribe the derivatives at 0 *only*, in addition to the function values at the above-mentioned nodes.

Interpolating polynomials of degree $2n-1$ based on the roots of nth Laguerre polynomials and the point 0 were introduced first by Egerváry and Turán [4] as the "most economical" stable interpolation on $[0, \infty)$. A convergence theorem was proved by Balázs and Turán [1] and later this process was investigated by Joó [7–10].

Lagrange interpolation for the Laguerre abscissas and its convergence were treated by Freud [5] and Nevai [11–13]. Let

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!} \{e^{-x} x^{n+\alpha}\}^{(n)}, \qquad n = 1, 2, \ldots,$$

be the Laguerre polynomial of degree n for $\alpha > -1$, with the usual normalization

$$L_n^{(\alpha)}(0) = \binom{n+\alpha}{n}.$$

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These polynomials are orthogonal on $[0, \infty)$ with respect to the weight function $e^{-x}x^{\alpha}$. The zeros of $L_n^{(\alpha)}(x)$ are

$$(0<)\, x_{1n}^{(\alpha)} < x_{2n}^{(\alpha)} < \cdots < x_{nn}^{(\alpha)}.$$

If there is no danger of misunderstanding we will write briefly x_{kn} or x_k, $k = 1, 2, \ldots, n$.

In what follows we will always suppose that α is *integer*. Let f be an α-times differentiable function on $[0, \infty)$. Let us denote by $Q_{n,\alpha}(f; x)$ its Hermite interpolating polynomial of degree $n+\alpha$ with nodes $x_{kn}^{(\alpha)}$, $k = 1, 2, \ldots, n$, and 0, the latter with multiplicity $\alpha + 1$. That is,

$$Q_{n,\alpha}(f; x) = \sum_{k=1}^{n} f(x_k)\left(\frac{x}{x_k}\right)^{\alpha+1} l_k(x) + \sum_{i=0}^{\alpha} f^{(i)}(0)\, r_i(x) \tag{1.1}$$

where $l_k(x)$ are the fundamental polynomials of Lagrange interpolation based on the roots of $L_n^{(\alpha)}(x)$:

$$l_k(x) = l_{kn\alpha}(x) = \frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)\prime}(x_k)(x - x_k)}, \qquad k = 1, 2, \ldots, n,$$

and the polynomials $r_i(x) = r_{in\alpha}(x)$ are such that

$$r_i^{(s)}(0) = \begin{cases} 1, & \text{if } s = i, \\ 0, & \text{if } 0 \leqslant s < i, \end{cases}$$

and

$$r_i(x_k) = 0, \qquad \text{for} \quad k = 1, 2, \ldots, n;\ i = 1, 2, \ldots, \alpha,$$

so that, explicitly,

$$r_i(x) = \frac{x^i L_n^{(\alpha)}(x)}{i!\binom{n+\alpha}{n}}, \qquad i = 0, 1, \ldots, \alpha.$$

In the case $\alpha = 0$ we have *Lagrange interpolation*:

$$Q_{n,0}(f; x) = \sum_{k=1}^{n} f(x_k)\frac{x}{x_k} l_k(x) + f(0)\, L_n^{(0)}(x). \tag{1.2}$$

Convergence theorems and estimates concerning $Q_{n,0}(f)$ were announced without proof by the author at the Varna Conference on Constructive Theory of Functions in 1984, [2].

We remark that convergence problems of Hermite interpolation of type $Q_{n,\alpha}$ based on the point 0 and Laguerre roots for non-integral α can be considered also, but these investigations require other means and will be treated in a forthcoming paper.

2. Results

We give weighted estimates which imply the convergence of interpolating polynomials $Q_{n,\alpha}(f)$ and their derivatives $Q_n^{(i)}(f)$ to f and $f^{(i)}$, respectively in $[0, \infty)$.

In what follows $O(1)$ is always independent from x and n. Our first theorem concerns Lagrange interpolation based on the roots of $L_n^{(0)}(x)$ and the origin (see (1.2)).

Theorem 1. *Let $f \in \operatorname{Lip} \gamma$, $\frac{1}{2} < \gamma \leqslant 1$, in $[0, \infty)$. Then*

$$|f(x) - Q_{n,0}(f; x)| = O(1)\, x^{1/2}\, e^{x/2} n^{-\gamma/2 + 1/4},$$

for $0 \leqslant x \leqslant x_{nn}^{(0)}$.

Note the important fact $x_{nn}^{(\alpha)} \sim n$ for the greatest zero of $L_n^{(\alpha)}(x)$, which follows from Lemma 3. We use the symbol $\sim$ in the sense of Szegő [14, p. 1]: if two sequences z_n and w_n of numbers have the property that $w_n \neq 0$ and the sequence $|z_n|/|w_n|$ has finite positive limits of indetermination, we write $z_n \sim w_n$.

Theorem 2. *Let $f^{(\alpha)} \in \operatorname{Lip} \gamma$, $0 < \gamma \leqslant 1$, in $[0, \infty)$ for some $\alpha > 0$ integer. Then*

$$|f(x) - Q_{n,\alpha}(f; x)| = O(1)\, x^{(\alpha + 1)/2}\, e^{x/2}\, n^{-(\alpha + \gamma)/2 + 1/4}$$

for $0 \leqslant x \leqslant x_{nn}$.

If $f^{(r)}$ exists for some $r > \alpha$, then we may have better estimates:

Theorem 3. *Let $f^{(r)} \in \operatorname{Lip} \gamma$, $0 < \gamma \leqslant 1$, in $[0, \infty)$ for some $r > \alpha$, where $\alpha \geqslant 0$ and integer. Then*

$$|f(x) - Q_{n,\alpha}(f; x)| = O(1)\, x^{(\alpha + 1)/2}\, e^{x/2}\, n^{-(r + \gamma)/2 + 1/4}$$

for $0 \leqslant x \leqslant x_{nn}^{(\alpha)}$.

Corollary. *The convergence of $Q_{n,\alpha}(f)$ to f is uniform in every finite subinterval of $[0, \infty)$ under the assumptions of the above theorems.*

THEOREM 4. *Suppose that* $f^{(r)}$ *exists in* $[0, \infty)$ *for some* $r \geqslant \alpha$, *where* $\alpha \geqslant 0$ *and integer. Let* $f^{(r)} \in \operatorname{Lip} \gamma$, $\frac{1}{2} < \gamma \leqslant 1$ *if* r *is even or* $f^{(r)} \in \operatorname{Lip} \gamma$, $0 < \gamma \leqslant 1$ *if* r *is odd. Then*

$$|f^{(i)}(x) - Q_{n,\alpha}^{(i)}(f; x)| = O(1)\, x^{(\alpha+1)/2 - i}\, e^{x}\, n^{-(r+\gamma)/2 + i + 1/4}$$

for $1 \leqslant i \leqslant [r/2]$ *and* $0 \leqslant x \leqslant x_{nn}^{(\alpha)}/2$.

COROLLARY. *The convergence of* $Q_{n,\alpha}^{(i)}(f)$ *to* $f^{(i)}$ *is uniform in every finite subinterval of* $[0, \infty)$ *if* $1 \leqslant i \leqslant [\alpha/2]$.

3. Lemmas and Proofs

LEMMA 1. *If* $f^{(r)}$ *exists and is continuous in* $[0, \infty)$, $r \geqslant 0$, *then there exists a polynomial* G_n *of degree* $n \geqslant 4r + 5$ *at most, that*

$$|f^{(i)}(x) - G_n^{(i)}(f; x)| = O(1)\, \omega\left(f^{(r)}; \frac{\sqrt{x(x_n - x)}}{n}\right)\left(\frac{\sqrt{x(x_n - x)}}{n}\right)^{r-i}$$

$$0 \leqslant x \leqslant x_n, \quad i = 0, 1, ..., r,$$

where $\omega(f^{(r)}; \cdot)$ *denotes the modulus of continuity of* $f^{(r)}$ *on* $[0, x_n]$.

The lemma shows that $G_n^{(i)}(f; 0) = f^{(i)}(0)$, $i = 0, 1, 2, ..., r$.

Proof. The lemma is an easy consequence of Gopengauz's theorem [6].

LEMMA 2 (Joó [10, inequality (11)]).

$$\frac{e^{x}}{x^{\alpha+1}} - \sum_{k=1}^{n} \frac{e^{x_k}}{x_k^{\alpha+1}} \left(\frac{L_n^{(\alpha)}(x)}{L_n^{(\alpha)\prime}(x_k)(x - x_k)}\right)^2 \geqslant 0, \qquad x > 0, \alpha > -1.$$

LEMMA 3. *Let* $\alpha > -1$. *Then the following asymptotic relation holds for the zeros* $x_k = x_{kn}^{(\alpha)}$ *of* $L_n^{(\alpha)}(x)$:

$$x_{kn}^{(\alpha)} \sim \frac{k^2}{n}, \qquad k = 1, 2, ..., n; n = 1, 2, ...$$

Proof. Lemma 3 follows from Theorem 6.31.3 of Szegő [14], e.g.,

LEMMA 4. *Let* $\alpha > -1$ *and* $\beta > \alpha/2 + \frac{1}{4}$. *Then for the zeros of* $L_n^{(\alpha)}(x)$ *the estimate*

$$\sum_{k=1}^{n} x_k^{\beta - \alpha - 1}(x_n - x_k)^{\beta}\, x^{\alpha+1}\, |l_k(x)| = O(1)\, n^{\beta + 1/4} x^{(\alpha+1)/2}\, e^{x/2}$$

holds for $x \geqslant 0$.

Proof. By Lemma 3 our sum is equal to

$$S_n = x_n^\beta \sum_{k=1}^{n} x_k^{\beta-\alpha-1}\left(1-\frac{x_k}{x_n}\right)^\beta x^{\alpha+1}\,|l_k(x)|$$

$$= O(1)\, n^\beta x^{(\alpha+1)/2} \sum_{k=1}^{n} x_k^{\beta-(\alpha+1)/2} e^{-x_k/2} e^{x_k/2}\left(\frac{x}{x_k}\right)^{(\alpha+1)/2} |l_k(x)|.$$

Using Cauchy's inequality and Lemma 2 we obtain

$$S_n = O(1)\, n^\beta x^{(\alpha+1)/2}\left\{\sum_{k=1}^{n} x_k^{2\beta-(\alpha+1)} e^{-x_k}\right\}^{1/2} e^{x/2}. \tag{3.1}$$

Let $-\frac{1}{2} < 2\beta - (\alpha+1) \leqslant 0$. Then denoting the sum under square root by T_n, we have by Lemma 3

$$T_n = \sum_{k=1}^{n} x_k^{2\beta-(\alpha+1)} e^{-x_k} = O(1)\sum_{k=1}^{n}\left(\frac{k^2}{n}\right)^{2\beta-(\alpha+1)} e^{-ck^2/n}$$

$$= O(1)\int_0^\infty \left(\frac{x^2}{n}\right)^{2\beta-(\alpha+1)} e^{-cx^2/n}\,dx = O(1)\, n^{1/2}, \tag{3.2}$$

where c is a positive constant.

In the case $2\beta - (\alpha+1) > 0$ the function $y(x) = (x^2/n)^{2\beta-(\alpha+1)} e^{-cx^2/n}$ $(x>0)$ attains its maximum at $x_0 = \sqrt{n(2\beta-(\alpha+1))/c}$, and y decreases monotonically, if $x > x_0$. Let $N = [x_0] + 1$, $N = O(1) n^{1/2}$ evidently. We get by repeated applications of Lemma 3,

$$T_n = \sum_{k=1}^{N} x_k^{2\beta-(\alpha+1)} e^{-x_k} + O(1)\sum_{k=N+1}^{n}\left(\frac{k^2}{n}\right)^{2\beta-(\alpha+1)} e^{-ck^2/n}$$

$$= O(1)\, N x_N^{2\beta-(\alpha+1)} + O(1)\int_N^\infty \left(\frac{x^2}{n}\right)^{2\beta-(\alpha+1)} e^{-cx^2/n}\,dx$$

$$= O(1)\, n^{1/2}. \tag{3.3}$$

The lemma follows from (3.1)–(3.3).

LEMMA 5 (Bernstein [3]). *Let* $M = \max_{0 \leqslant x \leqslant A} |P_n(x)|$, *where* $P_n(x)$ *is a polynomial of degree* n, *then*

$$|P_n^{(k)}(x)| \leqslant \left(\frac{k}{x(A-x)}\right)^{k/2} n^k M, \qquad k = 1, 2, \ldots, n;\ 0 \leqslant x \leqslant A.$$

Proofs of Theorems 1, 2, *and* 3. Only the proof of Theorem 3 $(r>\alpha)$ will be detailed, since the proofs of Theorems 2 and 1 can be treated as analog cases where $r=\alpha>0$ and $r=\alpha=0$, respectively.

Let $G_{n+\alpha}(f)$ be the polynomial defined in Lemma 1. Then we may write by Lemma 1,

$$\begin{aligned}|f(x)-Q_{n,\alpha}(f;x)| &\leqslant |f(x)-G_{n+\alpha}(f;x)|+|G_{n+\alpha}(f;x)-Q_{n,\alpha}(f;x)|\\ &=O(1)\,\omega\left(f^{(r)};\frac{\sqrt{x(x_n-x)}}{n}\right)\left(\frac{\sqrt{x(x_n-x)}}{n}\right)^r\\ &\quad+|Q_{n,\alpha}(G_{n+\alpha}f-f;x)|\end{aligned}$$

where $\omega(f^{(r)};\cdot)$ denotes the modulus of continuity of $f^{(r)}$ in $[0,\infty)$.

Using Lemma 3 and Lemma 1 again we get

$$\begin{aligned}&|f(x)-Q_{n,\alpha}(f;x)|\\ &\quad=O(1)\,x^{(r+\gamma)/2}n^{-(r+\gamma)/2}\\ &\quad\quad+O(1)\sum_{k=1}^{n}\omega\left(f^{(r)};\frac{\sqrt{x_k(x_n-x_k)}}{n}\right)\left(\frac{\sqrt{x_k(x_n-x_k)}}{n}\right)^r\\ &\quad\quad\times\left(\frac{x}{x_k}\right)^{\alpha+1}|l_k(x)|.\end{aligned}$$

Applying Lemma 4 $(\beta=(r+\gamma)/2)$ we obtain our theorem.

Proof of Theorem 4. Let $G_{n+\alpha}(f)$ be the polynomial defined in Lemma 1. Then we have by that lemma,

$$\begin{aligned}&|f^{(i)}(x)-Q_{n,\alpha}^{(i)}(f;x)|\\ &\quad\leqslant|f^{(i)}(x)-G_{n+\alpha}^{(i)}(f;x)|+|G_{n+\alpha}^{(i)}(f;x)-Q_{n,\alpha}^{(i)}(f)|\\ &\quad=O(1)\,\omega\left(f^{(r)};\frac{\sqrt{x(x_n-x)}}{n}\right)\left(\frac{\sqrt{x(x_n-x)}}{n}\right)^{r-i}+|Q_{n,\alpha}^{(i)}(G_{n+\alpha}f-f;x)|\end{aligned}$$

where $\omega(f^{(r)};\cdot)$ denotes the modulus of continuity of $f^{(r)}$ in $[0,\infty)$.

Applying Lemma 3, Lemma 5 for $Q_{n,\alpha}(f)$ if $A=2x$, and Lemma 1 again, we get

$$\begin{aligned}&|f^{(i)}(x)-Q_{n,\alpha}^{(i)}(f;x)|\\ &\quad=O(1)\,x^{(\gamma+r-i)/2}\,n^{-(\gamma+r-i)/2}\\ &\quad\quad+i^{i/2}x^{-i}n^{i}\max_{0\leqslant t\leqslant 2x}|Q_{n,\alpha}(G_{n+\alpha}f-f;t)|\end{aligned}$$

$$= O(1)\, x^{(\gamma + r - t)/2} n^{-(\gamma + r - t)/2}$$

$$+ O(1)\, x^{-t} n^{t} \max_{0 \leqslant t \leqslant 2x} \sum_{k=1}^{n} \omega\left(f^{(r)}; \frac{\sqrt{x_k(x_n - x_k)}}{n}\right)$$

$$\times \left(\frac{\sqrt{x_k(x_n - x_k)}}{n}\right)^{r} \left(\frac{t}{x_k}\right)^{\alpha + 1} |l_k(t)|.$$

Using Lemma 4 ($\beta = (\gamma + r)/2$) we can estimate the maximum of the last sum by

$$O(1)\, n^{-(\gamma + r)/2 + 1/4} x^{(\alpha + 1)/2} e^{x},$$

which proves the theorem.

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